

PROOF OF $e^{i\theta} = \cos \theta + i \sin \theta$

This proof was discovered by Euler in 1748. Today it is indispensable in topics such as AC circuits and damped harmonic motion.

We know:

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

Therefore,

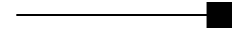
$$e^{i\theta} = \frac{(i\theta)^0}{0!} + \frac{(i\theta)^1}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \dots$$

Since $i = \sqrt{-1}$,

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right) + i \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right) \end{aligned}$$

And we know that $1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots = \cos \theta$ and $\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots = \sin \theta$. Therefore,

$$e^{i\theta} = \cos \theta + i \sin \theta$$



HERE'S SOMETHING ELSE ÜBER COOL

The five most important numbers in mathematics in one equation:

$$e^{i\pi} + 1 = 0$$

Source

Mr. "Why-Math-is-My-Life" McMaster, Personal Communication, June 2004